Gaps, almost disjoint families and a Ramsey ultrafilter

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Work in progress.

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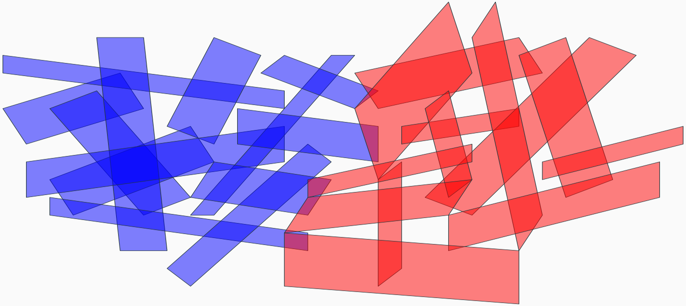
Pregap(in $\mathscr{P}(\omega)$)

Let $\mathcal{L}, \mathcal{R} \subseteq \mathscr{P}(\omega)$. We say that $(\mathcal{L}, \mathcal{R})$ is a pregap if

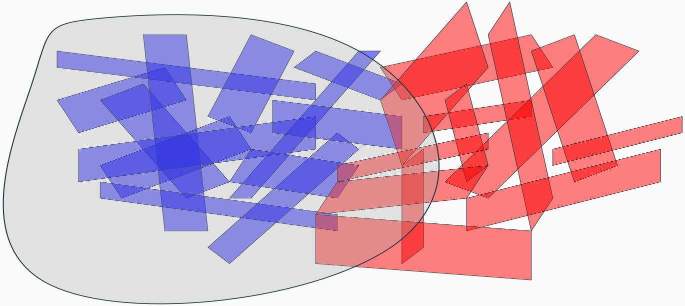
$$L \cap R =^* \emptyset \text{ for all } L \in \mathcal{L} \text{ and } R \in \mathcal{R}.$$



Gap(in $\mathscr{P}(\omega)$)

A pregap $(\mathcal{L}, \mathcal{R})$ is a gap, if there is NO $C \subseteq \omega$ such that:

$$L \subseteq^* C \text{ and } R \cap C^* = \emptyset \text{ for all } L \in \mathcal{L} \text{ and } R \in \mathcal{R}$$



Type of a pregap

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Let $(X, <_X)$ and $(Y, <_Y)$ be two partial orders. We say that a pregap $(\mathcal{L}, \mathcal{R})$ is an (X, Y)-pregap if $(\mathcal{L}, \subseteq^*) \equiv (X, <_X)$ and $(\mathcal{R}, \subseteq^*) \equiv (Y, <_Y)$.

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$\mathfrak{b}$ is the minimum cardinal for which there is an $(\omega, \mathfrak{b})$-gap.

Theorem(Baumgartner-Under PFA)

Let $\kappa \leq \lambda$ be infinite cardinals. There is a (κ, λ)-gap if and only if $(\kappa, \lambda) \in \{(\omega_1, \omega_1), (\omega, \mathfrak{b})\}$.

Why are gaps important?

According to Sikorski's extension Theorem, gaps in Boolean algebras are the only obstructions when we want to extend homomorphisms from one Boolean algebra to another one. For the particular case of gaps in ω, this is important since...

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Are (ω_1, ω_1) and $(\omega, \mathfrak{b})$ gaps really the only gaps definable in *ZFC*?

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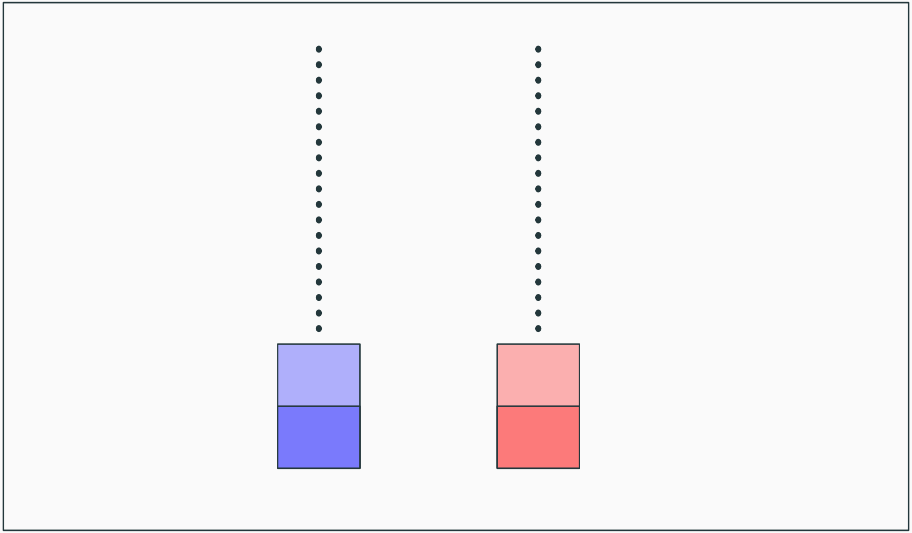
We will analyze this question for the case of (X, Y)-gaps where X and Y are partial orders of size ω_1.

How do (ω_1, ω_1)-pregaps look like?



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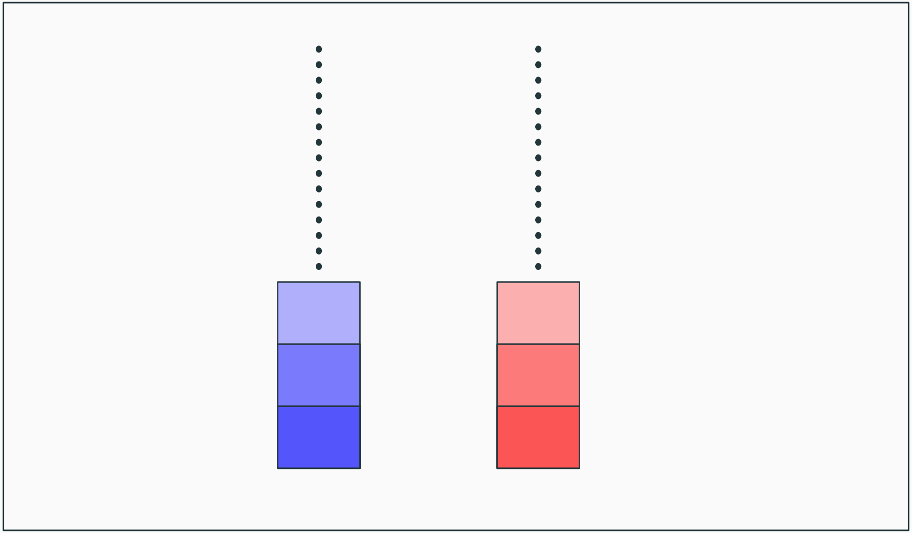

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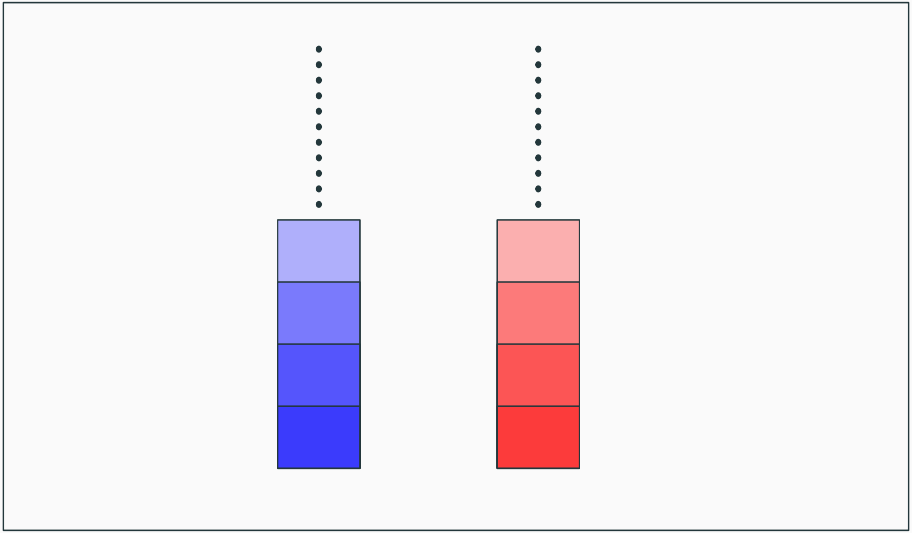

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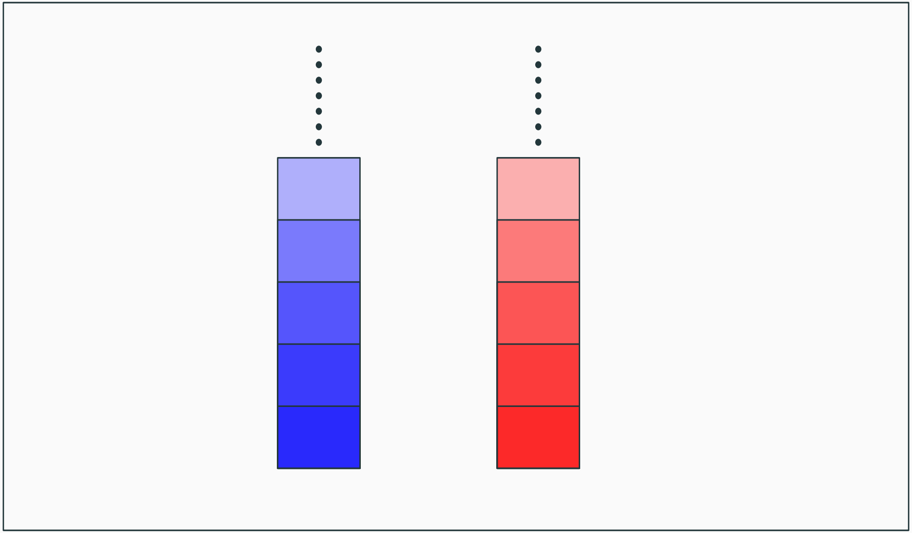

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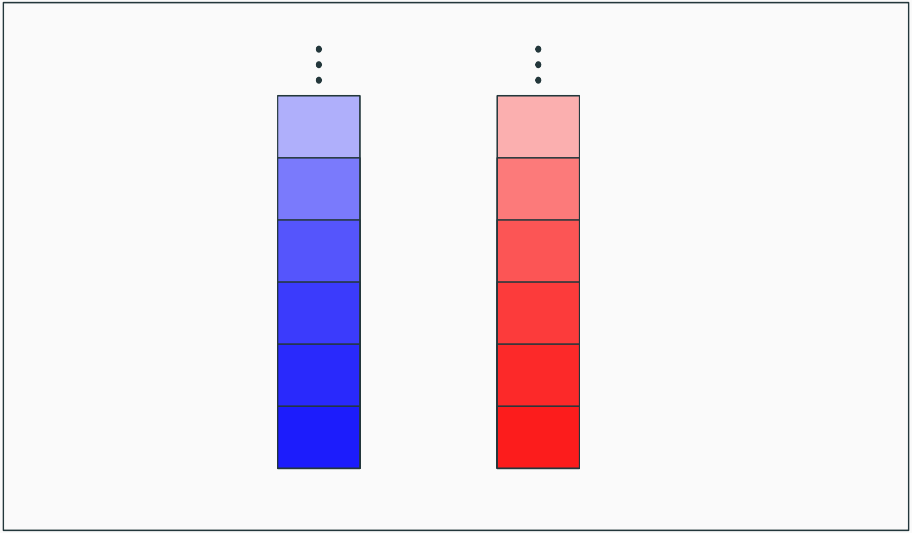
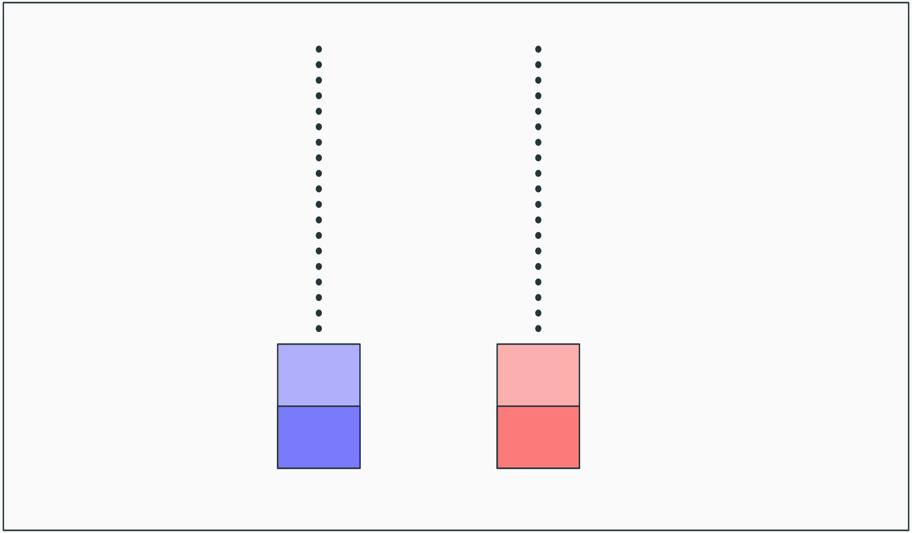


Oversimplifying the process



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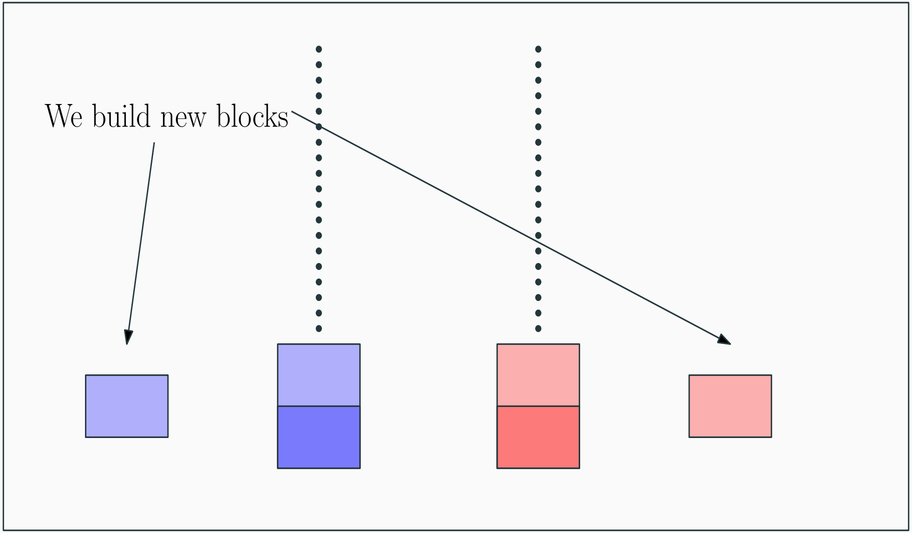

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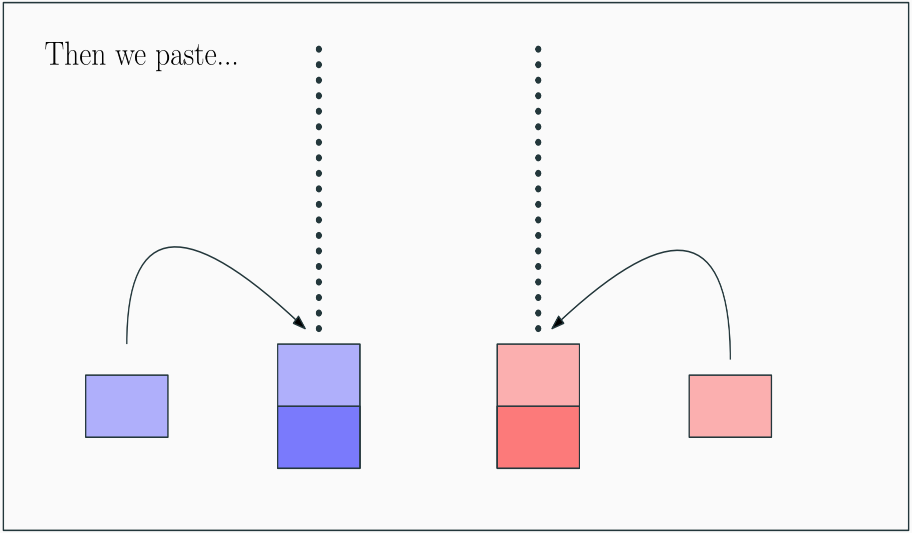
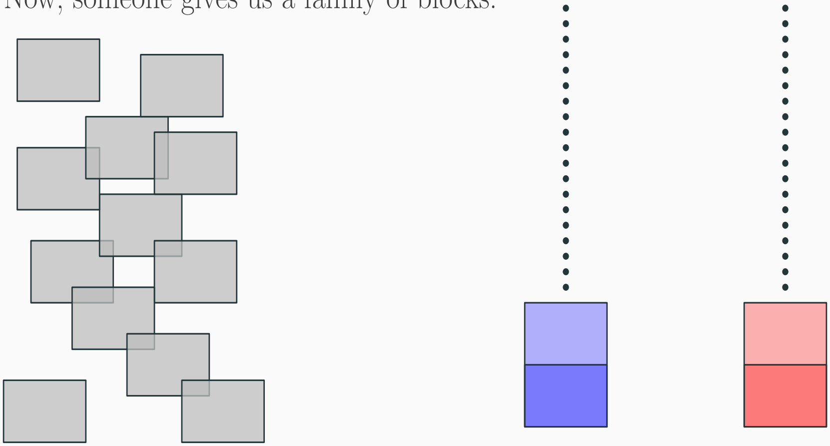


But what if..

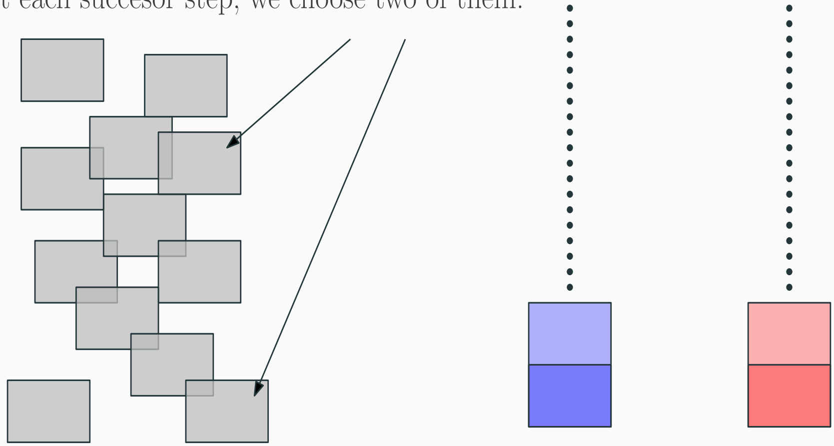
What if instead of building the “blocks”, we grab some existing ones?

Now, someone gives us a family of blocks.



rethinking the process

At each succesor step, we choose two of them.



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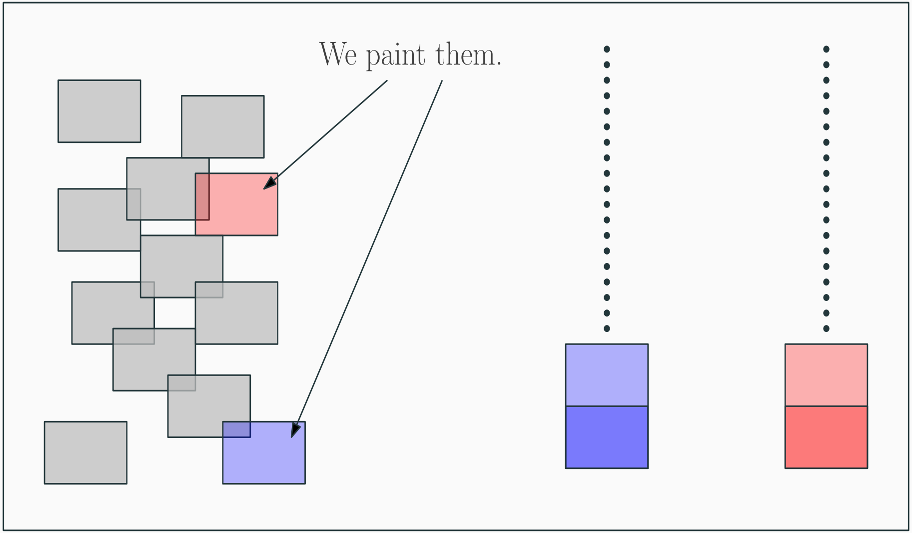

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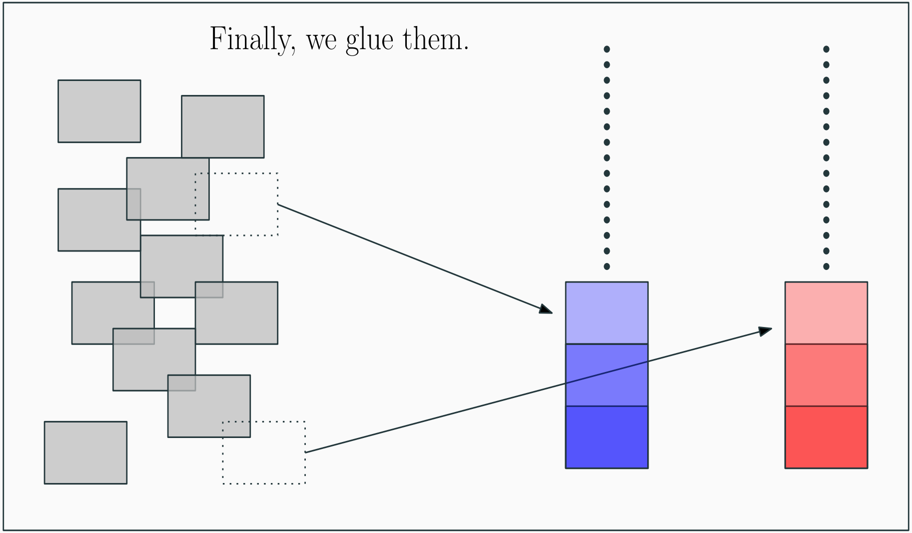

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Luzin families

An almost disjoint family (AD family) $\mathcal{A}$ is Luzin if $|\mathcal{A}| = \omega_1$ and for all disjoint $\mathcal{D}, \mathcal{E} \in [\mathcal{A}]^{\omega_1}$, the pair $(\mathcal{D}, \mathcal{E})$ is a gap.

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Theorem(Luzin)

There is a Luzin family.

Luzin representation

Let $(X, \leq)$ be a partial order of size ω_1. A *Luzin representation of X* is an ordered pair $(\mathcal{T}, \mathcal{A})$ of two families of infinite subsets of ω indexed as $\langle T_x \rangle_{x \in X}$ and $\langle A_x \rangle_{x \in X}$ respectively. Moreover, $\mathcal{A}$ is Luzin and the following conditions hold:

- $A_x \subseteq T_x$.
- If $y \not\leq x$ then $A_y \subseteq^* T_y \setminus T_x$.
- If $\inf(x, y)$ exists then $T_x \cap T_y =^* T_{\inf(x,y)}$.
- If $(\leftarrow, y) = \{z \in X \,:\, z < y\}$ has a maximum and equals x then $T_y \setminus T_x = A_y$.
- If there is no $z \in X$ with $z \leq x$ and $z \leq y$ then $T_x \cap T_y =^* \emptyset$.

If $\mathcal{A}$ is Luzin family and there is $\mathcal{T}$ so that $(\mathcal{T}, \mathcal{A})$ is a Luzin representation of X, we say that $\mathcal{A}$ codes X.

Definition

We say that a partial order $(X, \leq)$ is ω_1-like if:

- $|X| = \omega_1$,
- X is well founded,
- $|(\leftarrow, x)| \leq \omega$ for each $x \in X$.

Particularly, this implies that $cof(X) = \omega_1$.

The main theorem of this talk

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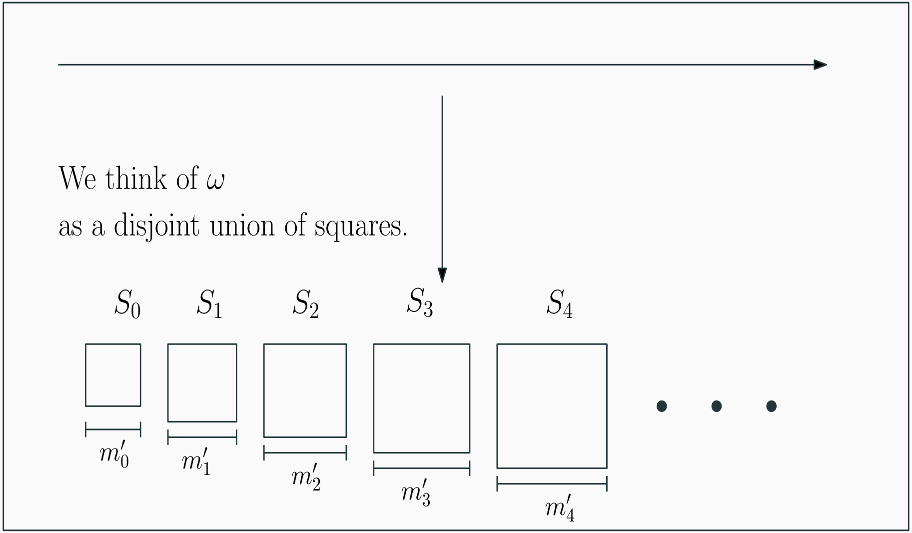
Theorem

There is a Luzin Family which codes any ω_1-like order.

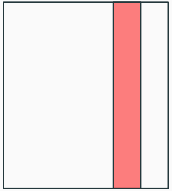
Describing the AD family from the main theorem



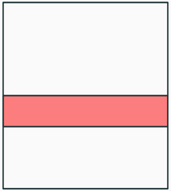
Describing the AD family of the main theorem

On each square S_n, an element A_α of $\mathcal{A}$ will either look like

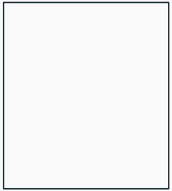
A vertical line in S_n.



An horizontal line in S_n



The empty set on S_n



Deciding horizontality verticality or emptyness

For each $\alpha \in \omega_1$, we decide whether $A_\alpha \cap S_n$ is an h. line, v. line or $\emptyset$ with a function

$$\Xi_\alpha : \omega \longrightarrow \{-1, 0, 1\}.$$

where -1 codes emptyness, 0 codes horizontality and 1 codes verticality.

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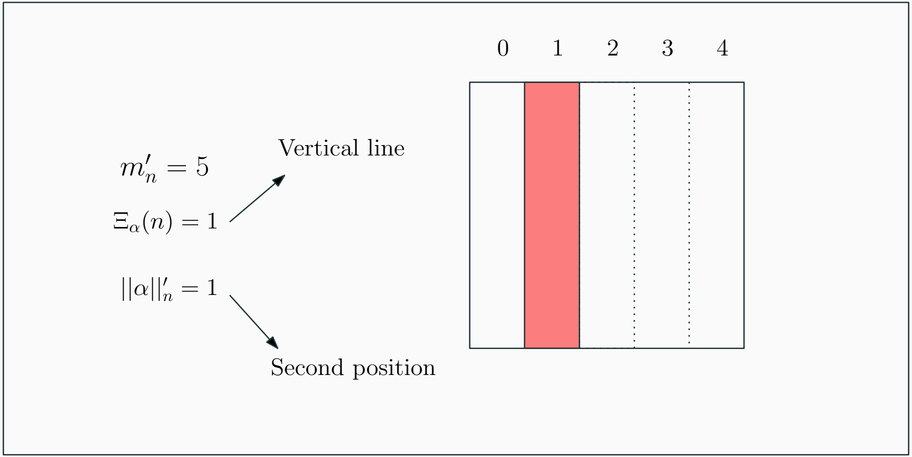
Deciding exact position of the line

To choose the exact position in the line (in case there is one), we use a function

$$\|\alpha\|'_{\_} : \omega \longrightarrow \omega.$$

so that $\|\alpha\|'_n < m'_n$ for each n (Recall that m'_n is the length of a side of the square).

Example


$m'_n = 5$
$\Xi_\alpha(n) = 1$
$||\alpha||'_n = 1$
Vertical line
Second position
0
1
2
3
4

What is missing...

A ρ-function

In order for the construction to work we need the existence of a function $\rho : [\omega_1]^2 \longrightarrow \omega$ which satisfies the following properties for each $\alpha < \beta \in \omega_1$:

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$\Xi_\alpha(n) = 0$

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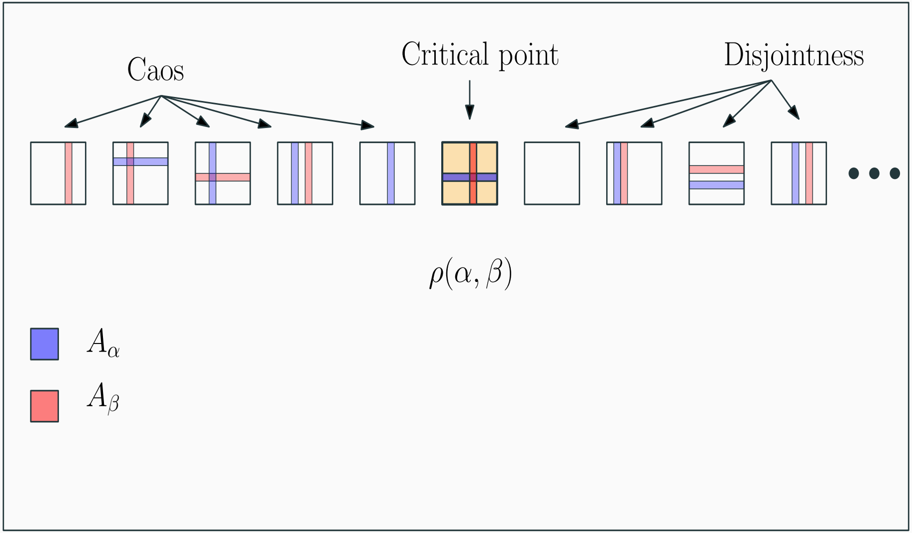
- Moreover ρ is an ordinal metric (ordinal metrics are functions reassembling metrics but with some minor (but important) differences).

The picture looks like this



In the actual construction, the funcions ρ, Ξ and $\|\alpha\|'_{-}$ are just coding a so called

$(\omega, 1)$-gap morass.

$(\omega, 1)$-gap morasses can be thought as a particular case of structures called construction schemes defined Todorcevic.

Corollary

Let X and Y be two ω_1-like orders. Then there is an (X, Y)-gap.

Proof.

An automatized fabric of gaps

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Consider $(\mathcal{T}, \mathcal{A})$ to be a Luzin representation of Z. Then $(\{T_x\}_{x \in X}, \{T_y\}_{y \in Y})$ is an (X, Y)-pregap.

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Note that if $C \subseteq \omega$ separates $(\{T_x\}_{x \in X}, \{T_y\})_{y \in Y}$, it would also separate $(\{A_x\}_{x \in X}, \{A_y\}_{y \in Y})$. But $\mathcal{A}$ is Luzin, so such C cannot exist. In other words, $(\{T_x\}_{x \in X}, \{T_y\})_{y \in Y}$ is a gap. □

It can be easily seen that any partial order of cofinality ω_1 has a cofinal ω_1-like subset. Therefore...

Corollary

For any two partial orders X and Y with $cof(X) = cof(Y) = \omega_1$, there are cofinal $X' \subseteq X$ and $Y' \subseteq Y$ so that there is an (X', Y')-gap.

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Corollary

There is an (ω_1, ω_1)-gap $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ so that the family

$$\{L_{\alpha+1} \backslash L_\alpha \ : \ \alpha \in \omega_1\} \cup \{R_{\alpha+1} \backslash R_\alpha \ : \ \alpha \in \omega_1\}.$$

is a Luzin family.

Is every (ω_1, ω_1)-gap, a gap due to an almost disjoint family?

Donut-inseparability

Definition

We say that an (ω_1, ω_1)-gap $(L_\alpha, R_\alpha)_{\alpha\in\omega_1}$ is donut-inseparable, if

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We have already seen that there are donut-inseparable gaps. However, there are also gaps which are not. This suggests the following definition.

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Definition

We say that an (ω_1, ω_1)-gap $(L_\alpha, R_\alpha)_{\alpha\in\omega_1}$ is weakly donut-inseparable if there is $X \in [\omega_1]^{\omega_1}$ so that $(L_\alpha, R_\alpha)_{\alpha\in X}$ is donut inseparable.

Gapness of (ω_1, ω_1)-gaps may be determined by *AD*-families

Theorem-Under *CH*

Any (ω_1, ω_1)-is weakly-donut inseparable.

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Theorem-Under *CH*

Any (ω_1, ω_1)-is weakly-donut inseparable.

Theorem

Suppose that $V \models CH$ and let κ be a cardinal. Then

$$\mathbb{C}_\kappa \Vdash \textit{Any } (\omega_1, \omega_1)\textit{-gap is weakly-donut separable.}$$

Thus, this statement is consistent with arbitrarily large continuum.

2-capturing construction schemes (as defined by Todorcevic) are particular subfamilies $\mathcal{F}$ of $[\omega_1]^{<\omega}$ which exist under $\Diamond$-principle. Their existence imply things as (Joint work with O. Guzman and S. Todorcevic):

- Strong L and S-spaces.
- Failure of Baumgartner's Axiom $BA(\omega_1)$.
- Suslin lattices in $\mathscr{P}(\omega)$.
- Suslin towers
- $\omega_1 \not\to (\omega_1, \omega + 2)^2$.
- A sixth Tukey type.

In particular, their existence is inconsistent with $MA + \neg CH$. This is unfortunate, as 2-capturing construction schemes can be used to define natural and useful *ccc*-forcings. Fortunately...

Definition

Let $\mathcal{F}$ be a 2-capturing construction scheme. We define

$\mathfrak{m}_{\mathcal{F}} = \min(\mathfrak{m}(\mathbb{P})\ :\ \mathbb{P}$ *is ccc and forces that* $\mathcal{F}$ *is 2-capturing* $)$

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Lemma

Consistently $\mathfrak{m}_{\mathcal{F}} > \omega_1$.

Theorem-Under $\mathfrak{m}_{\mathcal{F}} > \omega_1$

There is an (ω_1, ω_1)-gap which is **not** weakly donut-inseparable.

Problem 1

Suppose that X is a partial order of cofinality $\mathfrak{b}$. Is there a cofinal $Y \subseteq X$ for which there exists an (ω, Y)-gap? (An almost equivalent question would be: What are the types of orders of unbounded families in $(\omega^\omega, <^*)$?)

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What can we say about PFA and (X, Y)-gaps when X and Y are not cardinals? In other words, how much can Baumgartner's theorem can be extended?

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Problem 3

By translating the definitions to Boolean algebras in general: Is there a Boolean algebra $\mathbb{B}$ which admits an (ω_1, ω_1)-gap but **not** an (X, Y)-gap for some X, Y ω_1-like orders? What about Parovichenko Algebras? If the answer to this question is positive, it motivates the study of classification of orders in terms of gaps.

Problem 4

Let T be an ω_1-Suslin tree and $(L_s, R_s)_{s\in T}$ be an (T, T)-gap. Under which conditions can we assure that for any B uncountable branch of S in some generic extension, the pregap $(L_s, R_s)_{s\in T}$ is an (ω_1, ω_1)-gap?

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For a Luzin family $\mathcal{A}$, let $Spect(\mathcal{A}) = \{(X, <) : \mathcal{A} \text{ codes } X\}$. Under $\mathfrak{b} = \omega_1$ there is a Luzin family $\mathcal{A}$ so that $\omega_1 \notin Spect(\mathcal{A})$. Is this provable from ZFC?

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Problem 6

How much different can be the spectrum of two distinct Luzin families?

Problem 7

In which cannonical Models are there (ω_1, ω_1)-gaps which are not weakly donut-inseparable? Sacks Model, Random Model, Miller model,..

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Problem 8

What is the relation between *MA*, *PFA* or *PID* and donut-inseparability? In particular, what can we say about destructiblity in this realm?